Homework 6
Geometry

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Proposition 0.1 (Exercise 4-6). *Let M be a non-empty smooth compact manifold. Then there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.*

Proof. Suppose there is a smooth submersion $F : M \to \mathbb{R}^k$. By Proposition 4.28, F is an open map, so $F(M)$ is open in $\mathbb{R}^k$. Since F is continuous, it maps compact sets to compact sets, so $F(M)$ is compact in $\mathbb{R}^k$, and hence it is closed and bounded. As $\mathbb{R}^k$ is connected, the only sets that are open and closed are $\emptyset$ and $\mathbb{R}^k$. M is non-empty so $F(M)$ is non-empty, and $F(M)$ is bounded so $F(M) \neq \mathbb{R}^k$. Thus $F(M)$ can be no subset of $\mathbb{R}^k$, so no such F exists. □

Note: Exercise 4-10 was not assigned, but 4-13 said to use it, so I'm including a solution.

Proposition 0.2 (Exercise 4-10). *Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the standard projection, and let $q = \pi|_{S^n}$. Then q is a smooth covering map.*

Proof. First, we claim that q is a proper map. Let $K \subset \mathbb{RP}^n$ be compact. As $\mathbb{RP}^n$ is Hausdorff, K is closed, so $q^{-1}(K)$ is closed in S^n. Since S^n is compact, any closed subset is compact, thus $q^{-1}(K)$ is compact, thus q is proper. Now we show that q is a local diffeomorphism. Define

$$S_i^+ = \{(x_1, \dots x_{n+1} \in S^n | x_i > 0\}$$
$$S_i^- = \{(x_1, \dots x_{n+1} \in S^n | x_i < 0\}$$

Then we claim that $q|_{S_i^+}$ is a diffeomorphism. It is bijective because each $[x] \in \mathbb{RP}^n$ has a unique representative on the open half-sphere, and it is smooth because it is a composition of q with a (smooth) inclusion map.

We also show that the inverse is also smooth. Let $(\widetilde{U}, \widetilde{\phi})$ be a standard coordinate chart in $\mathbb{RP}^n$ (defined in Example 1.5 of Lee) and (U, ϕ) be a coordinate chart in S_i^+ (we can take $U = S_i^+$ and ϕ to be the projection $(x_1, \dots x_{n+1}) \mapsto (x_1, \dots \widehat{x}_i, \dots, x_{n+1})$). Then the coordinate representation of $q|_{S_i^+}^{-1}$ is

$$\begin{aligned}
\phi \circ (q|_{S_i^+})^{-1} \circ \widetilde{\phi}^{-1}(u_1, \dots u_n) &= \phi \circ (q|_{S_i^+})^{-1}[u_1, \dots u_{i-1} 1, u_i, \dots, u_n] \\
&= \phi(u_1, \dots u_{i-1}, 1, u_i, \dots u_n) \\
&= (u_1, \dots, u_n)
\end{aligned}$$

which is the identity map, and clearly smooth. Thus $q|_{S_i^+}$ is a diffeomorphism. By a similar argument, $q|_{S_i^-}$ is a diffeomorphism. Thus q is a local diffeomorphism, since every $p \in S^n$ is in some neighborhood $S_i^{\pm}$. Then by Proposition 4.46, every proper local diffeomorphism between nonemtpy connected manifolds is a smooth covering map, so q is a smooth covering map. □

Proposition 0.3 (Exercise 4-13). *Let $F : S^2 \to \mathbb{R}^4$ be the map $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.*

Proof. We define $\widetilde{F} : \mathbb{RP}^2 \to \mathbb{R}^4$ by $\widetilde{F} \circ q(v) = F(v)$. First we need to show that this is well-defined. (Note that q is surjective, we can define $\widetilde{F}$ just on the image of q as we have done.) Suppose $q(v), q(w) \in \mathbb{RP}^2$ such that $q(v) = q(w)$. Then $v = w$ or $v = -w$, and if $v = (x, y, z)$ we have

$$F(-v) = F(-x, -y, -z) = (x^2 - y^2, xy, xz, yz) = F(v)$$

so then $F(v) = F(w)$. Thus $\widetilde{F}$ is well-defined. Now we show that $\widetilde{F}$ is a smooth embedding. On some neighborhood, q is a diffeomorphism, so on that neighborhood $\widetilde{F} = F \circ q^{-1}$, so $\widetilde{F}$ is locally smooth, hence it is smooth. (I'm not sure how to do the rest, sorry.) □

Proposition 0.4 (Exercise 7-2). *Let G be Lie group, and let $m : G \times G \to G$ be the multiplication map $(g, h) \mapsto gh$ and let $i : G \to G$ be the inversion map $g \mapsto g^{-1}$. The differential $m_* : T_eG \oplus T_eG \to T_eG$ at the identity is given by*

$$m_*(X, Y) = X + Y$$

and the differential $i_* : T_eG \to T_eG$ is given by

$$i_*(X) = -X$$

Proof. Let $X, Y \in T_eG$. By linearity of the differential,

$$m_*(X, Y) = m_*((X, 0) + (0, Y)) = m_*(X, 0) + m_*(0, Y)$$

so we just need to compute $m_*(X, 0)$ and $m_*(0, Y)$. Let $\gamma : (-\epsilon, \epsilon) \to G$ be a curve with $\gamma(0) = e$ and $\gamma'(0) = X$. Then

$$m_*(X, 0) = \frac{d}{dt} m(\gamma(t), \gamma(0))\bigg|_{t=0} = \frac{d}{dt}(\gamma(t) * e)\bigg|_{t=0} = \gamma'(0) = X$$

and likewise if α is a curve with $\alpha(0) = e$ and $\alpha'(0) = Y$, we can do the same computation to get $m_*(0, Y) = Y$. Thus

$$m_*(X, Y) = m_*(X, 0) + m_*(0, Y) = X + Y$$

Now we compute the differential of the inversion map at the identity. Let $\gamma : (-\epsilon, \epsilon)$ be a curve with $\gamma(0) = e$ and $\gamma'(0) = X$. We have

$$e = m(\gamma(t), i \circ \gamma(t))$$

so

$$0 = m_*(X, i_*X) = X + i_*X \implies i_*X = -X$$

□

Proposition 0.5 (for Exercise 7-14)**.** *$U(n)$ is Lie subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *of dimension* n^2*.*

Proof. Define a smooth map $\phi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{M}(n, \mathbb{C})$ by $A \mapsto A^*A$ (where A^* is the conjugate transpose. Then $U(n)$ is the level set $\phi^{-1}(I_n)$. Let $\mathrm{GL}(n, \mathbb{C})$ act on itself by right multiplication, that is, $(A, B) \mapsto A \cdot B = AB$. We also define a right action $\mathrm{M}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \to \mathrm{M}(n, \mathbb{C})$ by

$$X \cdot A = A^*XA$$

And ϕ is equivariant with respect to these two actions, as we see:

$$\phi(A \cdot B) = \phi(AB) = (AB)^*AB = B^*A^*AB = B^*\phi(A)B = \phi(A) \cdot B$$

Note that the action of $\mathrm{GL}(n, \mathbb{C})$ on itself by right multiplication is transitive, so by the Equivariant Rank Theorem (Theorem 7.25 in Lee), F has constant rank. Thus by Theorem 5.12, the level set $\phi^{-1}(I_n)$ is a properly embedded submanifold with codimension equal to the rank of ϕ, so $U(n)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Now we compute the rank of ϕ. It has constant rank, so we just compute the rank at I_n. Let $\gamma : (-\epsilon, \epsilon) \to \mathrm{GL}(n, \mathbb{C})$ be the smooth curve $\gamma(t) = I_n + tB$ for some $B \in \mathrm{GL}(n, \mathbb{C})$. Then

$$\begin{aligned}
\phi_*(B) &= \frac{d}{dt}(\phi \circ \gamma)(t)\bigg|_{t=0} \\
&= \frac{d}{dt}(I_n + tB)^*(I_n + tB)\bigg|_{t=0} \\
&= \frac{d}{dt}(I_n^*I_n + tB^*I_n + I_n^*tB + t^2B^*B)\bigg|_{t=0} \\
&= \frac{d}{dt}(I_n + tB^* + tB + t^2B^*B)\bigg|_{t=0} \\
&= (B^* + B + 2tB^*B)|_{t=0} \\
&= B^* + B
\end{aligned}$$

Thus the image of ϕ_* is a subset of the space of Hermitian matrices. Conversely, if B is a Hermitian matrix, then $\phi_*\left(\frac{1}{2}B\right) = B$ so B is in the image of ϕ_*. Thus the image of ϕ_* is precisely the set of Hermitian matrices. While the Hermitian matrices do not form a vector space over $\mathbb{C}$, they do form a vector space of dimension n^2 over $\mathbb{R}$. Thus the image of ϕ_* has dimension at least n^2, but since $\mathrm{M}(n, \mathbb{C})$ has dimension n^2, the rank of ϕ cannot exceed n^2. Thus $U(n)$ is a Lie subgroup of codimension zero. □

Proposition 0.6 (Exercise 7-14)**.** *For $n \geq 1$, $SU(n)$ is a properly embedded $(n^2 - 1)$-dimensional Lie subgroup of $U(n)$.*

Proof. Let $\det : U(n) \to S^1$ be the smooth determinant map. It maps into S^1 because the determinant of a unitary matrix is a unit complex number. It is a homomorphism, so it is Lie group homomorphism, with $SU(n)$ as its kernel. Thus by Proposition. 7.16, $SU(n)$ is a properly embedded Lie subgroup of $U(n)$.

The detminant map is also surjective: If we modify the identity matrix to have e^{it} in the top right corner, then this modified matrix is a special unitary matrix with determinant e^{it}.

Thus by the Global Rank Theorem (Theorem 4.12), det is a smooth submersion. Thus it has rank 1, so again using Proposition 7.16, $SU(n)$ has dimension $n^2 - 1$. □

Lemma 0.7 (for Exercise 6)**.** *Let X be a smooth vector field on a smooth manifold M and let $\gamma : \mathbb{R} \to M$ be a nonnonstant periodic integral curve for X. That is, there exists $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for $t \in \mathbb{R}$. Then there exists T' such that $\gamma(s) = \gamma(t)$ if and only if $s - t = kT'$ for some $k \in \mathbb{Z}$. (We call T' the period of γ.)*

Proof. Consider the set $\mathcal{T} = \{T \in \mathbb{R} : \gamma(t+T) = \gamma(t) \text{ for all } t \in \mathbb{R}\}$. We claim that $\mathcal{T}$ is closed. Suppose that $x \in \mathbb{R} \setminus \mathcal{T}$ (if there is no x, then $0 \in \mathcal{T}$ and γ is constant). If for every $\epsilon > 0$, there exists $T \in \mathcal{T} \cap B(x, \epsilon)$, so we have $\gamma(t+T) = \gamma(t)$ with $T < \epsilon$. This implies that γ is a constant curve, since if γ were non-constant, it would only differ from $\gamma(t)$ on intervals of arbitrarily small width. Thus there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset \mathbb{R} \setminus \mathcal{T}$. Thus $\mathbb{R} \setminus \mathcal{T}$ is open, so $\mathcal{T}$ is closed.

Now let $T' = \inf \mathcal{T}$. Since $\mathcal{T}$ is closed, it contains its infimum, so $T' \in \mathcal{T}$. Since γ is nonconstant, $T' \neq 0$. Suppose that $s - t = kT'$ for some $k \in \mathbb{Z}$. Then

$$\gamma(s) = \gamma(t + kT') = \gamma(t + (k-1)T') = \ldots \gamma(t)$$

if k is positive, and a similar argument holds if k is negative. Conversely, suppose that $\gamma(s) = \gamma(t)$. Then

$$\{s' \in \mathbb{R} : \gamma(s) = \gamma(s')\} = \{s + kT' : k \in \mathbb{Z}\}$$

since T' is the infimum over all possible periods of γ. Thus $t = s + kT' \implies t - s = kT'$ for some $k \in \mathbb{Z}$. □

Proposition 0.8 (Exercise 6)**.** *Let X be a smooth vector field on a smooth manifold M and let $\gamma : I \to M$ be a nonconstant integral curve of X, where I is an open interval of $\mathbb{R}$. Then γ is a smooth immersion, and if γ is not injective, then there exists a smooth embedding $\phi : S^1 \to M$ and $c > 0$ such that $\gamma(t) = \phi(e^{ict})$.*

Proof. First we show that γ is an immersion. Let θ be the flow generated by X. Since γ is non-constant, there is some $t_0 \in I$ such that $\gamma'(t_0) = X_{\gamma(t_0)} \neq 0$, so $\gamma(t_0)$ is a regular point. Then by Proposition 9.21, $\theta^{\gamma(t_0)}$ is a a smooth immersion. As $\theta^{\gamma(t_0)}$ is an integral curve that coincides with γ at the point $\gamma(t_0)$, by uniqueness we have $\theta^{\gamma(t_0)} = \gamma$, hence γ is an immersion.

Now suppose that γ is not injective. Then there exist $s, t \in I$ such that $\gamma(s) = \gamma(t)$ and $s \neq t$. Let T be the period of γ, as defined in the previous lemma, and let $c = 2\pi/T$. Then define $\phi : S^1 \to M$ by $\phi(e^{it}) = \gamma(t/c)$. Then we have

$$\phi(e^{ict}) = \gamma(tc/c) = \gamma(t)$$

We claim that ϕ is injective. Let $e^{it_1}, e^{it_2} \in S^1$ such that $\phi(e^{it_1}) = \phi(e^{it_2})$. Then

$$\gamma(t_1/c) = \gamma(t_2/c) \implies t_1/c = t_2/c + kT \implies t_1 = t_2 + 2\pi k \implies e^{it_1} = e^{it_2 + 2\pi k} = e^{it_2}$$

Thus ϕ is injective. It is also an immersion, since it is a composition of the immersions $t \mapsto t/c$ and γ. Since S^1 is compact, by Proposition 4.22 we have that ϕ is a smooth embedding. □